Lecture 1

Learning Objectives

- ☐ To use set notations
- ☐ To apply operations (union, intersection) on sets
- ☐ To define de Morgan's Laws for sets
- ☐ To define relations on sets
- ☐ To define set partitions

RMIT University, Taylor's University

Set Theory

- Set is a collection of objects
- The { } notation for sets

- Example:
  X = { x | x is the alphabetical letter of English }

$$X = \{a, b, c, d, \ldots, x, y, z\}$$

$$a \in X \qquad b \in X \qquad p \in X \qquad z \in X$$

The element of Set

- The objects are called the elements of a set
- We use $\in$ to denote the elements of a set.

$$x \in A$$

x is an element of the set A

x is a member of A

x belongs to A

The element of Set

- The size of a set is called cardinality.
- The notation of cardinality is $|\quad|$.
- Example:

X = { x | x is the alphabetical letter of English }

X = { a, b, c, …, x, y, z }

$|X| = 26$

26 is the size or cardinality of X

$$Y = \{a, b\}, |Y| = 2$$

$$Z = \{b\}, |Z| = 1$$

A singleton (a 1-element set)

The Null Set / Empty Set

ϕ $\{\}$

Examples:

1. The set of numbers x such that $x^2 = -1$.

2. The set of students who are doing Industrial Training and taking MATH 2111.

Infinite Sets

- The set of all nonnegative integers
  - N = {1, 2, 3, …} (often used)
  - N = {0, 1, 2, 3, …} (in Maurer & Ralston)
- The set of all integers
  - Z = {…, -2, -1, 0, 1, 2, …}
- The set of all rational numbers
  - Q (quotients of integers or “fractions”)

Countably infinite
or
Countable

- The set of all real numbers, *R*

Uncountably infinite or **Uncountable**

Notations of Set

$$a \in X$$

$$|X| = 26$$

$$X \subseteq Y$$

$$X \subset Y$$

X is contained in *Y*

X is a subset of *Y*

Every element of *X* is also in *Y*.

Example

1. $X = \{1, 2, 3, 6, 7\}$
2. $Y = \{4, 5, 6, 7, 8\}$
3. $Z = \{2, 3, 6, 7\}$

$$Z \subseteq X \qquad\qquad Z \not\subset Y$$

More on Subsets

$$\phi \subseteq N \subseteq Z \subseteq Q \subseteq \Re \subseteq C$$

Note that the null set is regarded as a subset of every set, including itself.

Example

Let $A = \{a, b, c\} = \{b, c, a\}$. List all subsets of A.

- Ø (the no-element subset)
- $\{a\}, \{b\}, \{c\}$ (the 1-element subsets)
- $\{a, b\}, \{a, c\}, \{b, c\}$ (the 2-element subsets)
- $\{a, b, c\}$ (the 3-element subset)

Power Sets

- The set of all subsets of a given set *X*

$$\wp(X)$$

- Example: $A = \{a, b, c\}$

$$\wp(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, A\}$$

$$|\wp(A)| = 8$$

Theorem

$$|\wp(A)| = 2^{|A|}$$

Operations on Sets

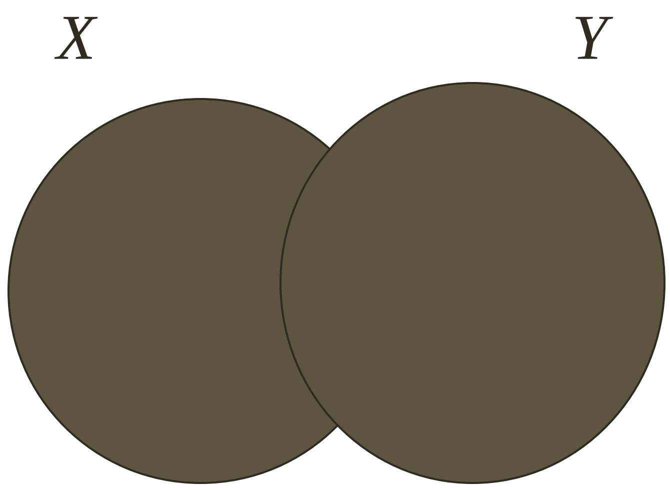
- The union of X and Y
  - The set of all elements in X <u>or</u> Y

$$X \cup Y$$

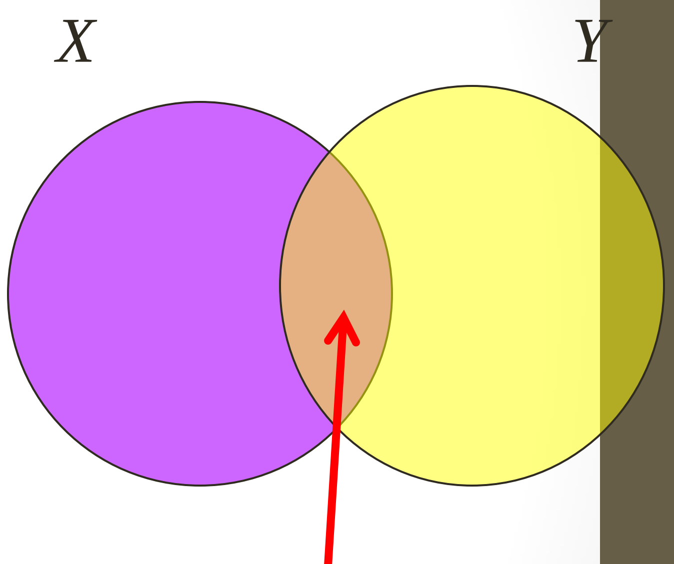
- The intersection of X and Y
  - The set of all elements in X <u>and</u> Y

$$X \cap Y$$

Venn Diagrams



$X \cup Y$

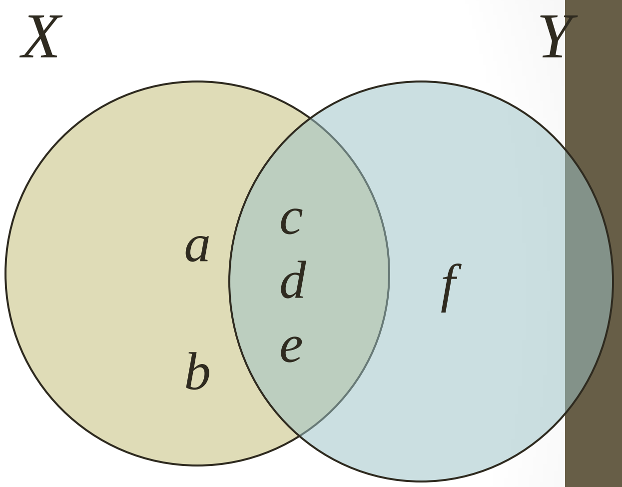


$X \cap Y$

Example

$$X = \{a, b, c, d, e\}$$

$$Y = \{c, d, e, f\}$$



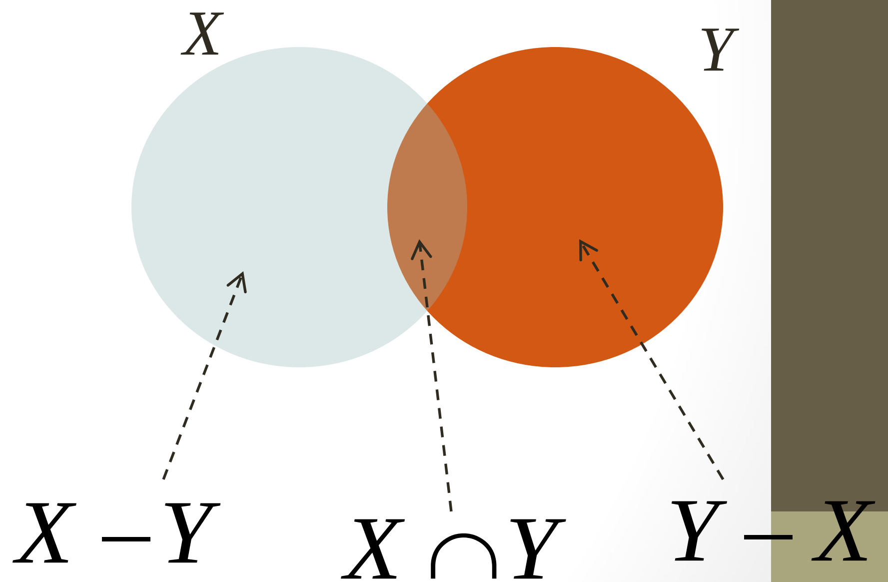
$$X \cup Y = \{a, b, c, d, e, f\}$$

$$X \cap Y = \{c, d, e\}$$

Complement of Y relative to X

- $X - Y$ or $X \setminus Y$
- The "set difference"

$$\{x \in X : x \notin Y\}$$



Disjoint Sets

- Two sets are disjoint if they don't intersect

$$X \cap Y = \phi \Rightarrow X \cup Y$$ is called the disjoint union of X and Y

denote

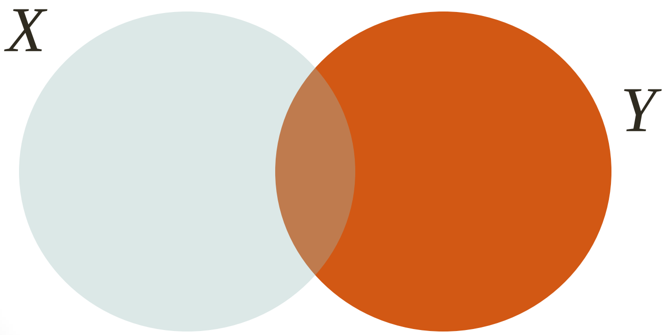
$$X \dot{\cup} Y$$

Disjoint Union of *X* and *Y*

$$X \cup Y = (X \setminus Y) \cup (X \cap Y) \cup (Y \setminus X)$$

$$= (X - Y) \cup (X \cap Y) \cup (Y - X)$$

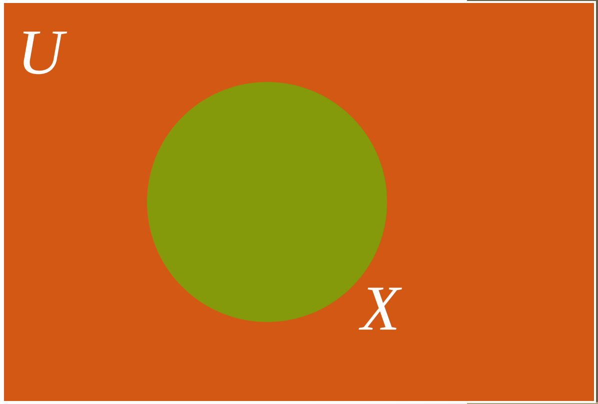
$$= [(X - Y) \dot{\cup} (Y - X)] \dot{\cup} (X \cap Y)$$



Universal Set

Often we have a universal set U consisting of all elements of interest.

So every other set of interest is a subset of U.



If $X \subseteq U$ we write

The complement of X

$$\overline{X} = U \setminus X$$
$$= U - X$$

Lemma: de Morgan's Law for Sets

$$\overline{X \cup Y} = \bar{X} \cap \bar{Y}$$

$$\overline{X \cap Y} = \bar{X} \cup \bar{Y}$$

Example

1. $U = \{1, 2, \ldots, 10\}$
2. $X = \{1, 2, 3, 4, 5\}$
3. $Y = \{2, 4, 6, 8\}$

Find:

$X \cup Y$ $\qquad \bar{X}$ $\qquad X - Y$

$X \cap Y$ $\qquad \bar{Y}$ $\qquad Y - X$

Examples of Some Other Sets

$$X = \{x \in \mathrm{Z} \mid x \geq 4\}$$

$$= \{x \in \mathrm{Z} : x \geq 4\}$$

$$= \{\text{all integers } \geq 4\}$$

$$= \{4, 5, 6, 7, \ldots\}$$

$$Y = \{x \in \Re \mid x^2 = 9\}$$

$$= \{-3, 3\}$$

Cardinality of Set Unions

For finite sets *X* and *Y*,

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

Relations on Sets

- Let X, Y be sets. A relation between X and Y is a subset of the Cartesian product
- Let R be the relation from X to Y

$$R = X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

- So a relation is a set of ordered pairs of the form (x, y), where $x \in X$ and $y \in Y$

Relations on Sets

Let ρ be a relation from x to y, and

$$(x, y) \in \rho$$

We write,

$$x\rho y$$

Read as "x rho y",
to say that "x is ρ-related to y"

Example 1

Let *R* be the relation from *X* to *Y*.

$$X = \{a, b, c\} \qquad Y = \{1,2\}$$

$$R = X \times Y = \{(x, y) \mid x \in X , y \in Y\}$$

$$= \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

Any subset of *R* is a relation from *X* to *Y*.

$$R_1 = \phi$$

$$R_2 = \{(a,1), (b,1), (c,1)\}$$

$$R_3 = \{(c,2)\}$$

… and 61 more

Example 2:

Let

$$X = \{3,4\} \qquad Y = \{3,4,5,6,7,8,9\}$$

If we define a relation *R* from *X* to *Y* by

$(x, y) \in R$ if *y* subtract *x* is a even number.

We obtain

$$R = \{(3,9),(4,8),(3,7),(4,6),(3,5),(4,4),(3,3)\}$$

The domain of *R* is $\{3,4\}$

The range of *R* is $\{3,4,5,6,7,8,9\}$

Relation on Sets

- When $X = Y$, a relation between X and Y is called a relation on X

$$X \times X = X^2 = \{(x, y) \mid x, y \in X\}$$

- Any subset of X^2 is a relation on X^2 .

Example 3

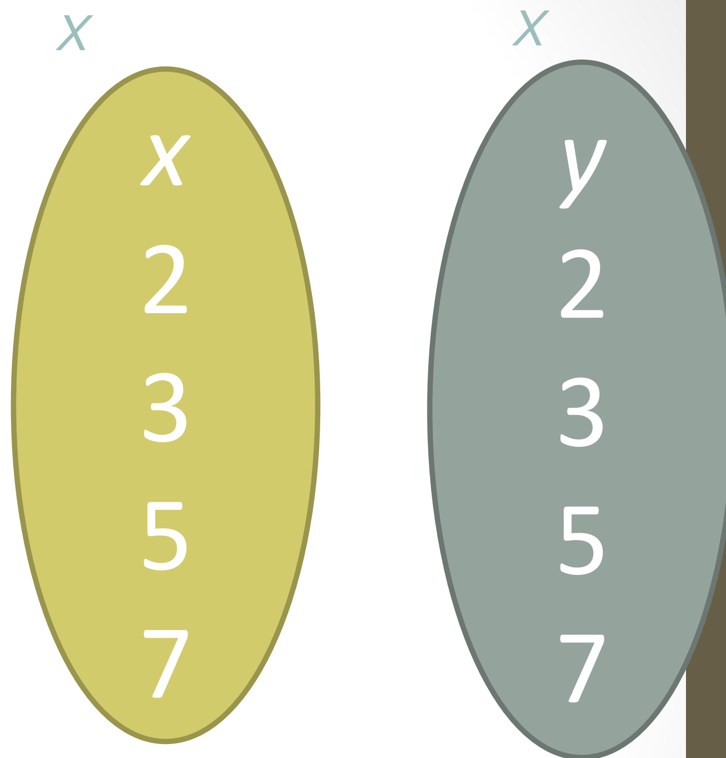
Let *R* be the relation on *X*

$$X = \{2,3,5,7\}$$

Define by

$$(x, y) \in R \quad \text{if} \quad x > y + 1$$

Then,



$$R = \{(x, y) \mid x > y + 1, x \in X, y \in X\}$$
$$= \{(5,2), (5,3), (7,2), (7,3), (7,5)\}$$

Properties of Relations: reflexive

Let ρ be a relation on X.

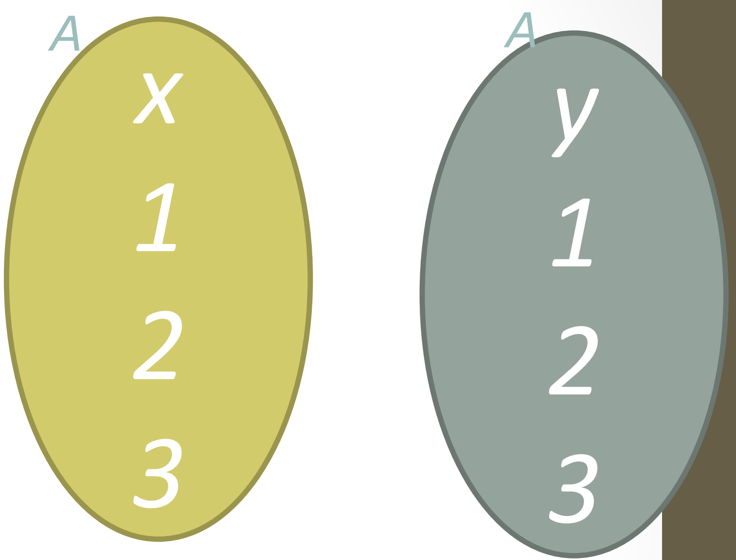
ρ is reflexive if

$$(x, x) \in \rho \qquad \text{for all } x \in X$$

Example 4:

Let $A = \{1, 2, 3\}$

and ρ be a relation on A defined as



$$x - y \geq 0$$

$$\rho = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$$

Therefore, ρ is reflexive

Properties of Relations: symmetric

Let ρ be a relation on X.

ρ is symmetric if $(x, y) \in \rho \Rightarrow (y, x) \in \rho$

Example 5:

Let $A = \{2, 3\}$

and ρ be a relation on A defined as

"$x \rho y$ if and only if $x + y$ is odd integer."

$\rho = \{(2,3),(3,2)\}$ ρ is symmetric

Properties of Relations: transitive

- ρ is transitive if for all

$$x, y, z \in X$$

If $(x, y) \in \rho$ and $(y, z) \in \rho$, and $(x, z) \in \rho$

Example 6:

Let $A = \{1, 3, 4\}$

and ρ be a relation on A defined as

"$x \rho y$ if and only if $\frac{x}{y} > 0$"

$$\rho = \{(1,1), (1,3), (1,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4)\}$$

Equivalence Relations

- A relation ρ on a set X is said to be an equivalence (relation) when it is reflexive, symmetric and transitive.
- Example 6 is a equivalence relation.

Congruence modulo *n*

- "a is congruent to b modulo n" when $(a - b)$ is an integer multiple of n

$$a \equiv b \pmod{n}$$

$$a - b = tn \qquad \text{for some integer } t$$

- Usually we want a and b to be integers

Example

$X = \{1, 2, 3, 4, 5, 6, 7\}$. Define ρ on X by $x \, \rho \, y$ if $x \equiv y \pmod{3}$. Write down ρ as a set of ordered pairs.

$\rho = \{(1,1), (1,4), (1,7), (2,2), (2,5), (3,3), (3,6), (4,1), (4,4), (4,7), (5,2), (5,5), (6,3), (6,6), (7,4), (7,7)\}$

Congruence modulo n

It can be shown that the relation $a \equiv b \pmod{n}$ is always an equivalence relation on Z and its subsets

I. $a \equiv a \pmod{n}$

II. $a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n}$

III. $$\left.\begin{matrix} a \equiv b \pmod{n} \\ b \equiv c \pmod{n} \end{matrix}\right\} a \equiv c \pmod{n}$$

Applications

- 4 o'clock + 12 hours = "1600 hours" = 16 o'clock = 4 o'clock
  - This is because $4 \equiv 16 \pmod{12}$
- 8 o'clock + 12 hours = "2000 hours" = 20 o'clock = 8 o'clock
  - This is because $8 \equiv 20 \pmod{12}$
- Coding theory is based on arithmetic modulo 2

Partitions

- Given a equivalence relation on a set X, we can partition X by grouping the related elements together.

- A partition is a set of disjoint, nonempty subsets of a given set X whose union is X

- Essentially, a partition divides X into subsets

Example 6 (revisited)

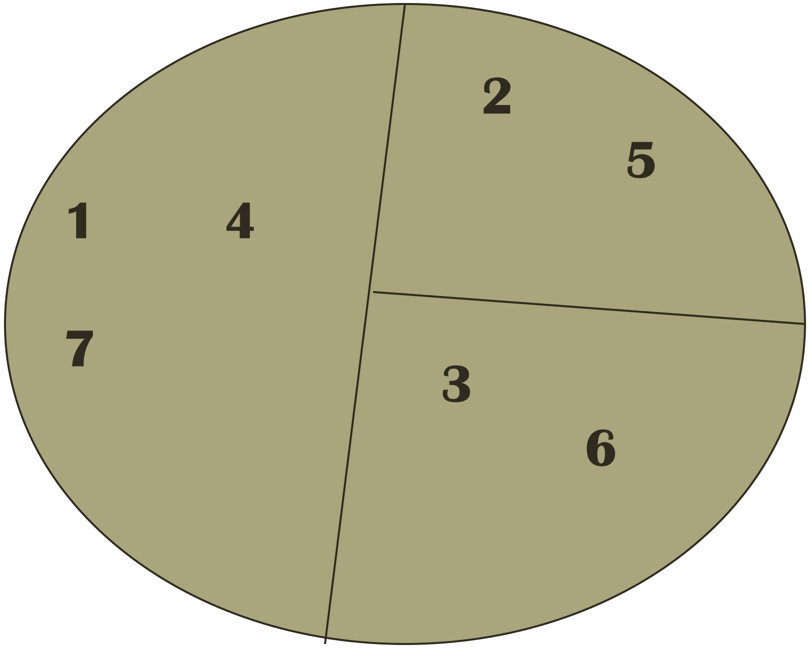
$X = \{1, 2, 3, 4, 5, 6, 7\}$. Define ρ on X by $x \rho y$ if $x \equiv y \pmod{3}$. Write down ρ as a set of ordered pairs.

$\rho = \{(1,1), (1,4), (1,7), (2,2), (2,5), (3,3), (3,6), (4,1), (4,4), (4,7), (5,2), (5,5), (6,3), (6,6), (7,4), (7,7)\}$

Theorem:
Equivalence classes of X given by the relation ρ.

For every equivalence relation there is a corresponding partition, and vice versa

Example



The partition corresponding to ρ is often denoted by Π_ρ.

Here: $\Pi_\rho = \{\{1,4,7\}, \{2,5\}, \{3,9\}\}$

Example

Consider the following collections of subsets of $S = \{1, 2, 3, \ldots, 8, 9\}$:

1. $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$
2. $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$
3. $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

Which of the above is a partition of S?

THE END